# Selling with Binding Reservations in the Presence of Strategic Consumers 

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## Agenda

- Motivation and research questions
- Related literature
- Model of selling with reservations
- Benchmark scenario: Price markdown
- Asymptotic approximation
- Revenue optimization, extensions and conclusions


## Motivation

## Current industry Practice

Retailers, airlines, entertainment industry recognize the need for price and demand segmentation

- Progressive markdowns
- Auctions
- Selling with reservations
- Withdrawable (relatively wide practice)
- Binding (limited practice)


## Academic Research: Revenue management

- Selling mechanisms: list prices, clearance seasons, auctions, non-binding reservations
- Discount policies: contingent, pre-announced
- Consumers' behavior: myopic, strategic


## Markdowns: Widespread practice



## Filene's

The price would be automatically reduced, first $25 \%$, then $50 \%$ and finally $75 \%$ !

## Empirical Facts

- In retail, $\approx 50 \%$ of items are sold at discount prices (Hardman, 2007)
- There are typically 2 selling seasons per year: spring-summer, fall-winter.
- Nakamura and Steisson (Q.J.Econ., 2008):
- Median duration of constant price period is 4.4-4.6 months
- Average duration of clearance seasons is 1.8-2.3 months


## Example: Selling with reservations - Sam's Club



## Sam's Club

Choose to Buy Later to purchase items automatically when they drop to that price, providing they do not sell out first.

Example provided by Elmaghraby et al. (2006)

## Example: Airtran's stand-by tickets



## Airtran

Pay for a segment, if the seat is available after the final boarding call, it's got your name all over it!

## Research questions

In industries like apparel, with high gross margin ( $\approx 38 \%$ ) and low net profits $(\approx 6 \%)$, small changes in revenue can have a big impact on financial performance.

## General

Is there any list price mechanism that combines the benefits of price discrimination, but does not have the drawbacks of the clearance season (e.g. display of merchandise at reduced prices, lower productivity of the shelf space)?

- Proposal: Selling with binding reservations.


## Sub-Questions

- How should the seller design such mechanism?
- How would rational consumers behave under this mechanism?
- What is the economic benefit of the mechanism?


## Literature

- Economics
- Coase (J. Law\&Econ., 1972), Fehr and Kuhn (J.P.Econ, 1995);
- DeGraba (RAND J. Econ., 1995).
- Revenue Management with strategic consumers
- Aviv et al. (2009).
- Elmaghraby et al. (MSOM, 2008), Liu and van Ryzin (MS, 2008);
- Yin et al. (MS, 2009), Alexandrov and Lariviere (2007)
- Closest: Elmaghraby et al. (POM, 2009. Single unit inventory, finite number of fixed and time-homogeneous valuations, no revenue optimization study).
- Methodology
- Maglaras and Meissner (MSOM, 2006); Caldentey and Vulcano (MS, 2007); Aviv and Pazgal (MSOM, 2008).


## Selling with reservations: Problem setup

- $\bar{Q}$ : seller's inventory endowment, with $Q_{0} \leq \bar{Q}$ units put up for sale;
- Finite selling horizon $[0, T]$;
- $p_{h}$ : regular product price during $[0, T)$, for purchases with an immediate delivery;
- $p_{l}$ : clearance price at time $T$ with $p_{l} \leq p_{h}$;
- $\lambda(t)$ : arrival rate of consumers (non-homogeneous Poisson);
- $v_{t}$ : valuation of a customer arriving at time $t, v_{t} \sim F(\cdot, t)$, independent, ( $K_{F}$-Lipschitz for all $t$, bounded support).
- At time $T$, leftover inventory is cleared among the consumers that placed reservations, according to a pre-announced strict time-based priority rule.


## Time based priority rules




Weak LIFO


Figure: Illustrations of strict (top) and weak (bottom) priority rules.

## IN THIS PRESENTATION:

- FIFO rationing rule, time-homogeneous valuations and arrival process.


## Consumer's problem

- Utility function:

$$
u(\tau, t, v-p)=(v-p) \exp (-w(t-\tau))
$$

where $\tau$ : arrival time, $t$ : time when a unit is received, $p$ : price paid, and $w$ : discount factor.

- Consumer makes a strategic decision based on $t, T, \lambda, Q_{0}, F$, and the rationing rule (Note: no real time information about $Q_{t}$, only $\mathbb{1}\left\{Q_{t} \geq 1\right\}$ is known).
- Stackelberg game (seller: leader; consumers: followers).
- Given a symmetric purchasing strategy $H$, consumer's decision is defined by:
- Buy now, with utility $u\left(\tau, \tau, v_{\tau}-p_{h}\right) \mathbb{P}\left(Q_{\tau} \geq 1 \mid H\right)$
- Reserve, with utility $u\left(\tau, T, v_{\tau}-p_{l}\right) \Pi_{H}(\tau)$
- Condition for placing a reservation for risk neutral consumers:

$$
u\left(\tau, T, v_{\tau}-p_{l}\right) \Pi_{H}(\tau) \geq u\left(\tau, \tau, v_{\tau}-p_{h}\right) \mathbb{P}\left(Q_{\tau} \geq 1 \mid H\right)
$$

## Consumer's decision: Threshold function $H(\tau)$



Figure: Sketch of strategy under the FIFO rationing rule when all consumers play according to a given $H(\tau)$.

## Question

Is there such function $H(\tau)$, which is an equilibrium strategy?

## Equilibrium analysis: FIFO

- Assume $v \in[0,1]$, and rescale parameters by setting $p_{l}=0$.
- Strategy space: $\mathcal{H} \subset \mathcal{D}$, the set of piecewise continuous functions with left and right limits.

$$
\mathcal{H}=\left\{H \in \mathcal{D}, H:[0, T] \rightarrow[0,1], \text { such that } H(t) \geq p_{h}, \text { for all } t\right\}
$$

- Condition for reservation: Reserve iff

$$
\frac{v_{\tau}}{v_{\tau}-p_{h}} \geq \frac{\exp (w(T-\tau)) \mathbb{P}\left(Q_{\tau} \geq 1 \mid H\right)}{\mathbb{P}\left(B\left(\Lambda_{H_{R}}(\tau)\right) \leq Q_{T}-1\right)} \triangleq g_{H}(\tau)
$$

- Proposition 1: Given a strategy $H$, a consumer $v_{\tau}$ places a reservation iff $v_{\tau} \leq \mathcal{R}(H)(\tau)$, where

$$
\mathcal{R}(H)(\tau) \triangleq \begin{cases}1 & \text { if } g_{H}(\tau) \leq \frac{1}{1-p_{H}} \\ \frac{p_{H} g_{H}(\tau)}{g_{H}(\tau)-1} & \text { if } g_{H}(\tau)>\frac{1}{1-p_{H}}\end{cases}
$$

- Notice, $R(H)(\tau)>p_{h}$ for all $\tau$.


## FIFO case: Properties of $\mathcal{R}(H)$

## Theorem 1

For any strict priority rationing rule, including FIFO, and for all strategies $H \in \mathcal{H}$, the best response $\mathcal{R}$ is a continuous mapping of $\mathcal{H} \rightarrow \mathcal{H}$, i.e.,

$$
\|\mathcal{R}(H)-\mathcal{R}(\tilde{H})\| \leq K_{H}\|H-\tilde{H}\|,
$$

for all $H \in \mathcal{H}$. Therefore, the set of strategies $\mathcal{H}$ exhibits the fixed point property, and an equilibrium strategy exists.
In addition, if $K_{H}<1$, then $\mathcal{R}$ is a contraction. In this case, the fixed point $\mathcal{R}\left(H^{*}\right)=H^{*}$ is guaranteed to be unique in $\mathcal{H}$ and can be found through the iteration $H^{n+1}=\mathcal{R}\left(H^{n}\right)$ starting from an arbitrary $H^{0} \in \mathcal{H}$.

Notation: We are denoting $B(a)$ a Poisson random variable with mean $a$, and $\beta(a) \triangleq \mathbb{P}(B(a)=a)$.

## Proof sketch

Idea: apply Schauder-Tychonoff fixed-point theorem
(1) Show that the set of feasible strategy profiles is $\mathcal{H}$ is a compact convex set.
(2) For any strict priority rationing rule defined by $\xi(\cdot)$, and for any strategy profile $H \in \mathcal{H}, \Pi_{H}(\tau)$ is differentiable and $\left|\Pi_{H}^{\prime}(\tau)\right| \leq K_{\Pi}<\infty$, for all $\tau$.
(3) For all $H \in \mathcal{H}$, there is a positive constant $K$ (independent of $H$ ) such that the best-response strategy $\mathcal{R}(H)(\tau)$ is a $K$-Lipschitz continuous function.
(4) Prove that the best-response $\mathcal{R}$ mapping is continuous in $\mathcal{H}$, i.e.,

$$
\|\mathcal{R}(H)-\mathcal{R}(\tilde{H})\| \leq K_{H}\|H-\tilde{H}\|
$$

for all $H \in \mathcal{H}$.

## Symmetric equilibrium strategy under FIFO



Figure: An example of a purchasing strategy $H(t)$ under the FIFO rationing rule for the case $w=T=1$, $\lambda=10$, and $p_{h}=0.5$, when $p_{l}$ is normalized to be zero and the valuations are Unif $[0,1]$. The dots represent a sample path of the arrival process. In this case, $Q_{0}=6$. Four consumers buy at the full price $p_{h}$; and seven place reservations. The two earliest reservations are allocated the two leftover units at time $T$. In this case, there are five unsuccessful reservations.

## Benchmark setup: Random allocation (RA-s) rationing rule

- Pre-announced fixed discount model of a clearance season analyzed by Aviv and Pazgal (MSOM 2008).
- Regular season $\left[0, T_{s}\right]$, and clearance season $\left(T_{s}, T\right]$, where $T_{s}=s T, s \in[0,1]$.
- Modeling consumer returns: Consumers can place reservations during [ $0, T_{s}$ ] which will be satisfied at $T_{s}$. Rationing is performed at random. Consumers continue to arrive during ( $T_{s}, T$ ].
- Key feature: probability of getting an item through a reservation is the same for all consumers

$$
\Pi_{H}(\tau)=\min \left\{\frac{Q_{T_{S}}}{\# \text { of consumers that reserved an item }}, 1\right\} \triangleq c(H)
$$

- Reservation condition:

$$
\frac{v_{\tau}}{v_{\tau}-p_{h}} \geq \frac{\exp \left(w\left(T_{S}-\tau\right)\right) \mathbb{P}\left(Q_{\tau} \geq 1 \mid H\right)}{c(H)} \triangleq g_{H}^{R A}(\tau)
$$

## RA-s properties

- There is an infimum $\tilde{v}>p_{h}$ such that $\mathcal{R}(H)(\tau) \geq \tilde{v}$, for all $H \in \mathcal{H}$.
- For all $H \in \mathcal{H}, \mathcal{R}(H)(\tau)$ is $K$-Lipschitz continuous.
- The set of strategies $\mathcal{H}$ equipped with the uniform norm $\|X\|=\sup _{0 \leq \tau \leq \tau_{s}}\{|X(\tau)|\}$ in $\left[0, T_{S}\right]$ exhibits the fixed-point property.
- PE exists


## Symmetric equilibrium strategy under RA-1



Figure: An example of a purchasing strategy $H(\tau)$ under the Random Allocation rationing rule, with parameters $w=T_{S}=1, \lambda=10, p_{h}=0.5, p_{I}=0$ and valuations Unif[0,1]. The dots represent a sample path of the arrival process. In this case, $Q_{0}=6$. Four consumers buy at the full price $p_{h}$; and seven place reservations. Two of the reservations are allocated the leftover units at time $T_{S}$. There are five unsuccessful reservations.

## Asymptotic analysis: FIFO case

Consider a sequence of problems indexed by $n$ such that

$$
\lim _{n \rightarrow \infty} \frac{\lambda^{n}}{n}=\lambda \quad \lim _{n \rightarrow \infty} \frac{Q_{0}^{n}}{n}=Q_{0} \quad \lim _{n \rightarrow \infty} \frac{Q_{0}^{n}}{\lambda^{n} T} \triangleq \rho=\frac{Q_{0}}{\lambda T} .
$$

## Theorem 2

Suppose that the purchasing strategy $H(\tau)$ is given. Then, in the limit as $n \rightarrow \infty$ :
(i) If we let $\Lambda_{H_{B}}(\tau)$ be the average number of "buy-nows" up to time $\tau$, then the re-scaled number of units $Q_{\tau}^{n} / n$ converges weakly to a constant $Q_{\tau} \triangleq\left(Q_{0}-\Lambda_{H_{B}}(\tau)\right)^{+}$.
(ii) If we let $\Lambda_{H_{R}}(\tau)$ be the average number of consumers that place reservations up to time $\tau$, then the probability $\mathbb{P}\left(B\left(\Lambda_{H_{R}}^{n}(\tau)\right) \leq Q_{T}^{n}-1\right)$ converges weakly to the distribution:

$$
F_{B\left(\Lambda_{H_{R}}(\tau)\right)}\left(Q_{T}\right)=\left\{\begin{array}{lll}
1 & \text { if } \Lambda_{H_{R}}(\tau)<Q_{T} \\
0 & \text { if } \Lambda_{H_{R}}(\tau)>Q_{T}
\end{array}\right.
$$

- Kev feature: Consumers are able to deduce the cuurrent inventory level


## Asymptotic strategies: FIFO

- Limited supply: $Q_{0} \leq \lambda T \bar{F}\left(p_{h}\right)$

$$
H(\tau)=\left\{\begin{array}{cl}
p_{h} & \text { if } \tau \leq \tau^{*}=\frac{Q_{0}}{\lambda \bar{F}\left(p_{h}\right)} \\
s(\tau) \in[0,1] & \text { else }
\end{array}\right.
$$

- Intermediate supply: $\lambda T \bar{F}\left(p_{h}\right) \leq Q_{0} \leq \lambda T$

$$
H^{*}(\tau)=\left\{\begin{array}{cl}
\min \left\{\frac{p_{h} \exp (w(T-\tau))}{\exp (w(T-\tau))-1}, 1\right\} & \text { if } \tau \in\left[0, \tau^{*}\right) \\
p_{h} & \text { if } \tau \in\left[\tau^{*}, T\right]
\end{array}\right.
$$

where $\tau^{*}=\frac{Q_{0}-\lambda T \bar{F}\left(p_{h}\right)}{\lambda F\left(p_{h}\right)}$ is the time of the last reservation placed and satisfied.

- Abundant supply: $Q_{0}>\lambda T$

$$
H^{*}(\tau)=\min \left\{\frac{p_{h} \exp (w(T-\tau))}{\exp (w(T-\tau))-1}, 1\right\} .
$$

## Accuracy of the approximation: FIFO

| $Q_{0}$ |  | Exact PE |  | Approx. PE |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Error |  |  |  |  |  |
|  | Res. | Buy-now | Res. | Buy-now |  |
| 7 | 6.98 | 3.02 | 6.56 | 3.44 | $8.40 \%$ |
| 14 | 13.93 | 6.07 | 13.13 | 6.87 | $8.08 \%$ |
| 35 | 34.26 | 15.74 | 32.82 | 17.18 | $5.79 \%$ |
| 70 | 67.73 | 32.27 | 65.63 | 34.37 | $4.21 \%$ |
| 140 | 134.60 | 65.40 | 131.26 | 68.74 | $3.34 \%$ |



Asymptotic performance for the FIFO rationing rule case, for valuations Unif[ $[0,1]$, $T=w=1, p_{h}=0.5$, and $\rho=0.7$.

## Asymptotic strategies: RA-s

## Theorem 3

Suppose that the purchasing strategy $H(\tau)$ is given. Then, in the limit as $n \rightarrow \infty$, the probability of getting an item after placing a reservation converges weakly to

$$
c^{\infty}(H) \triangleq \min \left\{\frac{\left(Q_{0}-\Lambda_{H_{B}}\left(T_{S}\right)\right)^{+}}{\Lambda_{H_{R}}\left(T_{S}\right)}, 1\right\}
$$

- Abundant supply: same as for FIFO
- Intermediate supply:

$$
\begin{gathered}
H^{*}(\tau)=\min \left\{\frac{p_{h} \exp \left(w\left(T_{S}-\tau\right)\right)}{\exp \left(w\left(T_{S}-\tau\right)\right)-c^{\infty}\left(H^{*}\right)}, 1\right\}, \\
c^{\infty}\left(H^{*}\right)=1-\frac{(1-\rho) T_{S}}{\int_{0}^{T_{S}} F\left(\min \left\{\frac{p_{h} \exp \left(w\left(T_{S}-\tau\right)\right)}{\exp \left(w\left(T_{S}-\tau\right)\right)-c^{\infty}\left(H^{*}\right)}, 1\right\}\right) \mathrm{d} \tau}
\end{gathered}
$$

- Solution to the fixed point equation $c^{\infty}(H)$ exists, possibly more than one.
- Limited supply: multiple equilibria possible with $c^{\infty}(H) \geq 0$.


## RA-s: Multiple equilibria

- $H_{2}(\tau)$ Pareto-dominates $H_{1}(\tau)$.
- Can be shown in general that the equilibrium with the highest value of $c^{\infty}(H)$ is Pareto dominant.


Two equilibrium purchasing strategies for the RA rationing rule in the asymptotic regime with intermediate supply and (scaled) valuations $\operatorname{Beta}(0.4,0.4), T_{S}=1, w=0$, $p_{h}=0.45$, and $1>\rho=0.55>\bar{F}\left(p_{h}\right) \approx 0.527$. Here, $c^{\infty}\left(H_{1}\right)=0.10$, and $c^{\infty}\left(H_{2}\right)=0.55$.

## Asymptotic analysis: RA-s

| $Q_{0}$ | Exact PE |  | Approx. PE |  | Error |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Res. | Buy-now | Res. | Buy-now |  |
| 7 | 7.65 | 2.35 | 8.37 | 1.63 | $14.25 \%$ |
| 14 | 15.85 | 4.15 | 16.73 | 3.27 | $8.81 \%$ |
| 35 | 41.06 | 8.94 | 41.83 | 8.17 | $3.07 \%$ |
| 70 | 83.42 | 16.58 | 83.65 | 16.35 | $0.46 \%$ |
| 140 | 167.05 | 32.95 | 167.30 | 32.70 | $0.25 \%$ |



Asymptotic approximation for the RA rationing rule for a case with valuations Unif $[0,1]$, $T_{S}=w=1, p_{h}=0.5$, and $\rho=0.7$.

## Seller's revenue optimization problem

- Seller's objective (all parameters are scaled back to the original ones)

$$
\begin{aligned}
V(\bar{Q}) & =\max _{T, Q_{0}, p_{l}, p_{h}}\left\{p_{h} \lambda \int_{0}^{T} e^{-\alpha t} \mathbb{1}\left\{Q_{t}>0\right\} \bar{F}(H(t)) \mathrm{d} t\right. \\
& \left.+p_{l} e^{-\alpha T} \min \left\{\left(Q_{0}-\Lambda_{H_{B}}(T)\right)^{+}, \Lambda_{H_{R}}(T)\right\}, \quad \text { subject to } p_{l} \leq p_{h}, Q_{0} \leq \bar{Q}\right\} .
\end{aligned}
$$

- Optimization can also be performed w.r.t. the rationing rules
- Revenue benchmark (RA-s)

$$
\begin{aligned}
V(\bar{Q}) & =\max _{T, Q_{0}, p_{l}, p_{h}}\left\{p_{h} \lambda \int_{0}^{T_{S}} e^{-\alpha t} \mathbb{\Perp}\left\{Q_{t}>0\right\} \bar{F}(H(t)) \mathrm{d} t\right. \\
& \left.+p_{l} e^{-\alpha T_{S}} \min \left\{\left(Q_{0}-\Lambda_{H_{B}}\left(T_{S}\right)\right)^{+}, \Lambda_{H_{R}}\left(T_{S}\right)\right\}+V(C), \text { subject to } p_{l} \leq p_{h}, Q_{0} \leq \bar{Q}\right\},
\end{aligned}
$$

where $V_{C} \triangleq \frac{1}{\alpha} p_{l} \bar{F}\left(p_{l}\right) \lambda\left(\exp \left(-\alpha T_{S}\right)-\exp \left(-\alpha \min \left\{T, \tau^{*}\right\}\right)\right)$, for $\alpha>0$, is the revenue collected during the clearance season, and $\tau^{*} \triangleq T_{S}+\left(Q_{0}-\lambda \bar{F}\left(p_{l}\right) T_{S}\right)^{+} /\left(\lambda \bar{F}\left(p_{l}\right)\right)$ is the purchasing time of the last available unit.

## Numerical results: Revenue gaps with respect to FIFO



Default values of parameters are: $Q_{0}=500, T=1, \lambda=1000, \alpha=0.5, w=2, v \sim \operatorname{Unif}[0,1]$ FIFO induces a higher number of "buy-nows", therefore higher revenues. Typically, items are sold at clearance prices $20-40 \%$ of the time (Nakamura and Steinsson, QJE, 2008). I.e., $s=0.6-0.8$.

## Structure of the revenue






Percentage price decrease (left), and fraction of transactions at low price (right), as a function of a) $\alpha$, b) $w$, and c) $Q_{0}$.

Default values of parameters:
$Q_{0}=500, T=1, \lambda=1000, \alpha=$ $0.5, w=2, b=1$.



## Extensions: Consumer surplus






Default values of parameters are: $Q_{0}=500, T=1, \lambda=1000, \alpha=0.5, w=2, v \sim \operatorname{Unif}[0,1]$ FIFO sells more units in total, therefore, delivers a higher consumer surplus.

## Extensions: Mixed market effects

- The arrival rate $\lambda$ is split between a fraction $\gamma$ of myopic and $1-\gamma$ strategic consumers.
- The myopic consumers play $H(\tau)=p_{h}$.
- It can be shown that, although, the exact strategies of the forward-looking consumers depend on $\gamma$, the dependence vanishes in the asymptotic regime.



Left: Relative revenue increase under FIFO, compared to RA-s. Right: Relative revenue loss from an erroneous assumption of myopic consumers' behavior as a function of $\gamma$. Value of parameters: $Q_{0}=500, T=1, \lambda=1000, \alpha=0.5, w=2$, and valuations Unif $[0,1]$.

## Concluding remarks

- We developed a stylized model where a seller operates a pricing with binding reservations scheme, and the consumers are strategic.
- We proved that an equilibrium always exists in the resulting game, and that it can be computed using an iterative algorithm.
- Asymptotic analysis provides a simple and accurate approximation to the problem. The purchasing behavior converges weakly to a an equilibrium that can be characterized in closed form.
- We observed that the pricing with reservations mechanism under the FIFO rationing rule dominates RA-s (including RA-1) when the seller is more patient than consumers and
- The supply-demand ratio $\rho \triangleq Q_{0} /(\lambda T)$ is moderate to large, and/or
- The dispersion of the consumers' valuations is moderate to high.
- The revenue gap between FIFO and the usual markdown practice can exceed $5 \%$.


## Thank you!

